J. Math. Anal. Appl. 358 (2009) 10–17

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Existence of finite-energy electroweak monopoles

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ARTICLE INFO

Article history:
Received 11 March 2009
Available online 22 April 2009
Submitted by J. Xiao

Keywords:
Electroweak monopoles
Dynamical shooting method
Existence and uniqueness
Asymptotic estimates

ABSTRACT

Finite-energy electroweak monopoles have important physical applications in the phenomenology of electroweak interaction. In this paper, we establish the existence and uniqueness of such monopoles by a dynamical shooting method and obtain sharp asymptotic estimates for the solutions.

1. Introduction

Ever since Dirac [4] first introduced the concept of magnetic monopoles, monopoles have remained a fascinating subject in theoretical physics. Monopoles in Abelian gauge theory have been generalized to those in non-Abelian gauge theory by Wu and Yang [10] who showed that the pure $SU(2)$ gauge theory allowed a point-like monopole, and by 't Hooft [8] and Polyakov [7] who have constructed a finite-energy monopole solution in Georgi–Glashow model as a topological soliton. In an earlier work, Gibbons et al. [5] and Lee and Weinberg [6] and Yang [11] showed that adding regularizing terms were important for achieving finite-energy magnetic monopoles. In the interesting case of the electroweak theory of Weinberg and Salam, however, it has generally been asserted that there exists no topological monopole of physical interest. The basis for this non-existence theorem is that with the spontaneous symmetry breaking the quotient space $SU(2) \times U(1)/U(1)_{em}$ allows no non-trivial second homotopy. This makes people think that there is no topological structure in Weinberg–Salam model which can accommodate a magnetic monopole. In fact, Cho and Maison [3] have established that the Weinberg–Salam model and Georgi–Glashow model have exactly the same topological structure, and demonstrated the existence of a new type of monopole and dyon solutions in the standard Weinberg–Salam model. Thus the Weinberg–Salam model does have the same non-trivial second homotopy as the Georgi–Glashow model which allows topological monopoles. So people can proceed to construct the desired monopole and dyon solutions in the Weinberg–Salam model. Originally Cho–Maison obtained the solutions by numerical simulation. A proof of existence of such solutions using a variational method was presented in [12,13].

The Cho–Maison solution carries an infinite energy at the classical level, which means that physically the mass of the monopole remains arbitrary. Consequently people wonder whether one can have an analytic electroweak monopole which has a *finite energy*. Cho and Kimm [2] showed this is indeed possible. The purpose of this paper here is to establish the existence and uniqueness of such a new monopole solution by a dynamical shooting method [9] and obtain sharp asymptotic estimates for the solutions.

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.04.034

In the second section, we briefly review the background of finite-energy electroweak monopoles. In the third section, we first discuss the mathematical structure of the existence problem. We then state our main existence and uniqueness theorem. In the fourth section, we transform the first-order finite-energy electroweak monopole equations into a second-order equation, and then introduce the Euler transformation to reduce the equation into a semilinear equation. The existence problem of an electroweak monopole is seen to be equivalent to the existence problem of a nonlinear two-point boundary value problem. Finally, we present a dynamical shooting method which solves the existence and uniqueness problem completely. We also obtain sharp asymptotic estimates for the solutions.

2. Finite-energy electroweak monopole problem

Following Cho and Kimm [2] and Bae and Cho [1], we recall that the Lagrangian describing the standard Weinberg–Salam model is given by

$$\begin{aligned} Ł &= -|\hat{D}_\mu \phi|^2 - \frac{\lambda}{2}\left(\phi^+\phi - \frac{\mu^2}{\lambda}\right)^2 - \frac{1}{4}(F_{\mu\nu})^2 - \frac{1}{4}(G_{\mu\nu})^2, \\ \hat{D}_\mu \phi &= \left(D_\mu - i\frac{g'}{2}B_\mu\right)\phi, \end{aligned} \tag{2.1}$$

where ϕ is the Higgs doublet, $F_{\mu\nu}$ and $G_{\mu\nu}$ are the gauge field strengths of $SU(2)$ and $U(1)$ with the potentials A_μ and B_μ, D_μ is the covariant derivative of $SU(2)$ subgroup defined in terms of the Pauli spin matrices $(\tau^a) = \tau$ by

$$D_\mu \phi = \left(\partial_\mu - \mathrm{i}\frac{g}{2}\tau \cdot A_\mu\right)\phi,$$

and g and g' are the corresponding coupling constants. From (2.1) one has the following equations of motion

$$\begin{cases} \hat{D}_\mu(\hat{D}_\mu\phi) = \lambda\left(\phi^+\phi - \dfrac{\mu^2}{\lambda}\right)\phi, \\ D_\mu F_{\mu\nu} = i\dfrac{g}{2}\left[\phi^+\tau(\hat{D}_\nu\phi) - (\hat{D}_\nu\phi)^+\tau\phi\right], \\ \partial_\mu G_{\mu\nu} = i\dfrac{g'}{2}\left[\phi^+(\hat{D}_\nu\phi) - (\hat{D}_\nu\phi)^+\phi\right]. \end{cases} \tag{2.2}$$

Now we follow Cho and Kimm [2] and Bae and Cho [1] to choose the following static spherically symmetric ansatz

$$\begin{cases} \phi = \dfrac{1}{\sqrt{2}}\rho(r)\xi(\theta,\phi), \\ \xi = i\begin{pmatrix} \sin(\frac{\theta}{2})q^{-i\phi} \\ -\cos(\frac{\theta}{2}) \end{pmatrix}, \\ \hat{\phi} = \xi^+\tau\xi = -\hat{r}, \\ A_\mu = \dfrac{1}{g}A(r)\hat{\phi}\partial_\mu t + \dfrac{1}{g}\left(f(r)-1\right)\hat{\phi}\times\partial_\mu\hat{\phi}, \\ B_\mu = -\dfrac{1}{g'}B(r)\partial_\mu t - \dfrac{1}{g'}(1-\cos\theta)\partial_\mu\phi, \end{cases} \tag{2.3}$$

where (t, r, θ, ϕ) are spherical coordinates and q and ρ denote the electric charge and the Higgs field. Equations of motion (2.2) reduce to following equations

$$\begin{cases} f'' - \dfrac{f^2-1}{r^2}f = \left(\dfrac{g^2}{4}\rho^2 - A^2\right)f, \\ \rho'' + \dfrac{2}{r}\rho' - \dfrac{f^2}{2r^2}\rho = -\dfrac{1}{4}(B-A)^2\rho + \lambda\left(\dfrac{\rho^2}{2} - \dfrac{\mu^2}{\lambda}\right)\rho, \\ A'' + \dfrac{2}{r}A' - \dfrac{2f^2}{r^2}A = \dfrac{g^2}{4}\rho^2(A-B), \\ B'' + \dfrac{2}{r}B' = \dfrac{g'^2}{4}\rho^2(B-A). \end{cases} \tag{2.4}$$

Notice that in the unitary gauge the Lagrangian (2.1) can be written as

$$\begin{aligned} Ł = &-\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{4}G_{\mu\nu}^2 - \frac{1}{2}|D_\mu W_\nu - D_\nu W_u|^2 + igF_{\mu\nu}W_\mu^* W_\nu + \frac{1}{4}g^2\left(W_\mu^* W_\nu - W_\nu^* W_\mu\right)^2 - \frac{1}{2}(\partial_\mu\rho)^2 \\ &- \frac{1}{4}\rho^2\left(g^2 W_\mu^* W_\nu + \frac{1}{2}(g'B_\mu - gA_\mu)^2\right) - \frac{\lambda}{2}\left(\frac{\rho^2}{2} - \frac{\mu^2}{\lambda}\right)^2. \end{aligned} \tag{2.5}$$

To regularize the Cho–Maison dyon, Cho and Kimm [2] and Bae and Cho [1] introduce an extra interaction Ł$_1$ to (2.5)

$$Ł_1 = i\alpha g F_{\mu\nu} W_\mu^* W_\nu + \frac{\beta}{4} g^2 \left(W_\mu^* W_\nu - W_\nu^* W_\mu\right)^2, \tag{2.6}$$

where α and β are arbitrary constants. So the energy of system is given by

$$E = E_0' + E_1,$$

where

$$E_0' = \frac{2\pi}{g^2} \int_0^\infty \frac{dr}{r^2} \left\{ \frac{g^2}{g'^2} + 1 - 2(1+\alpha) f^2 + (1+\beta) f^4 \right\}.$$

In order to make the energy finite, we are required to have

$$\begin{cases} \dfrac{g^2}{g'^2} + 1 - 2(1+\alpha) f^2(0) + (1+\beta) f^4(0) = 0, \\ (1+\alpha) f(0) - (1+\beta) f^3(0) = 0. \end{cases}$$

Thus, we arrive at the following condition for a finite-energy solution

$$\begin{cases} 1+\beta = (1+\alpha)^2 \sin^2\theta_w = (1+\alpha)^2 \dfrac{q^2}{g^2}, \\ f(0) = \dfrac{1}{\sqrt{(1+\alpha)\sin^2\theta_w}}, \end{cases}$$

where θ_w is the classical Weinberg mixing angle. However, this condition does not guarantee the smoothness of the gauge potentials at the origin. Cho and Kimm [2] and Bae and Cho [1] impose the condition $f(0) = 1$ and obtain the energy lower bound for the monopole

$$E \geqslant \left| \frac{1}{q} \int d^3x\, \partial_i \left(\frac{g}{2} \varepsilon_{ijk} F_{jk\rho} \right) \right|.$$

Furthermore this bound is saturated by the solutions of the following equations

$$\begin{cases} \varepsilon_{ijk} D_j W_k \pm i(1+\alpha) q \rho W_i = 0, \\ \partial_i \rho \mp \varepsilon_{ijk} \left[i(1+\alpha) q W_j^* W_k - \dfrac{g}{2q} F_{jk} \right] = 0, \end{cases} \tag{2.7}$$

which is very similar to the well-known Bogomol'nyi–Prasad–Sommerfield monopole equation of Georgi–Glashow model. In fact, the first-order differential equations are the first-order finite-energy electroweak monopole equations and much more tractable than the second-order field equations, and also solve it easily.

3. Mathematical structure and existence theorem

Notice that in the unitary gauge the spherically symmetric ansatz (2.3) is written as

$$\begin{cases} \rho = \rho(r), \\ W_\mu = \dfrac{i}{g} \dfrac{f(r)}{\sqrt{2}} q^{i\phi} (\partial_\mu \theta + i \sin\theta\, \partial_\mu \phi), \\ A_\mu = -\dfrac{1}{g} A(r) \partial_\mu t - \dfrac{1}{g} (1 - \cos\theta) \partial_\mu \phi, \\ B_\mu = -\dfrac{1}{g'} B(r) (1 - \cos\theta) \partial_\mu \phi. \end{cases} \tag{3.1}$$

Inserting the ansatz (3.1) into (2.7), we obtain the following equations

$$\begin{cases} f'(r) \pm q(1+\alpha) \rho(r) f(r) = 0, \\ \rho'(r) \mp \dfrac{1}{qr^2} \left(1 - (1+\alpha) \sin^2\theta_w f^2(r)\right) = 0 \end{cases} \tag{3.2}$$

with the boundary conditions

$$f(0) = 1, \qquad f(\infty) = 0, \qquad \rho(0) = 0, \qquad \rho(\infty) = \rho_0 > 0.$$

It is important to note that Eqs. (3.2) are actually the first-order finite-energy electroweak monopole equations (2.7) under the ansatz (3.1).

Now we consider the generalized form of the first-order finite-energy electroweak monopole equations (3.2)

$$\begin{cases} f'(r) \pm qa\rho(r) f(r) = 0, \\ \rho'(r) \mp \dfrac{1}{qr^2}\big(1 - bf^2(r)\big) = 0, \end{cases} \tag{3.3}$$

where $a > 0$, $b > 0$, $q > 0$. In this paper, choosing $a \neq b = 1$ and making a change $(f(r), q\rho(r)) \to (f(r), \rho(r))$, we consider the following equations

$$\begin{cases} f'(r) \pm a\rho(r) f(r) = 0, \\ \rho'(r) \mp \dfrac{1}{r^2}\big(1 - f^2(r)\big) = 0. \end{cases} \tag{3.4}$$

Without loss of generality, we consider the upper sign in (3.4), because the lower sign case may be obtained from the upper sign case after a change of dependent variables, $(f(r), \rho(r)) \to (f(r), -\rho(r))$. In this case, we are to find a solution of the boundary value problem

$$\begin{cases} f'(r) + a\rho(r) f(r) = 0, \quad 0 < r < \infty, \\ \rho'(r) - \dfrac{1}{r^2}\big(1 - f^2(r)\big) = 0, \quad 0 < r < \infty, \\ f(0) = 1, \qquad f(\infty) = 0, \qquad \rho(0) = 0, \qquad \rho(\infty) = q\rho_0 > 0. \end{cases} \tag{3.5}$$

Our main existence and uniqueness theorem for finite-energy electroweak monopoles can be stated as follows.

Theorem 3.1. *For any real number $a > 0$, the two-point boundary value problem* (3.5) *has a unique solution $(f(r), \rho(r))$, so that $f(r)$ is strictly decreasing and $\rho(r)$ is strictly increasing for $r > 0$. Besides, there hold the sharp asymptotic estimates*

$$f(r) = 1 + O\big(r^{\frac{1+\sqrt{8a+1}}{2}}\big), \qquad \rho(r) = O\big(r^{\frac{-1+\sqrt{8a+1}}{2}}\big) \quad \text{as } r \to 0,$$

$$f(r) = O\big(e^{-aq\rho_0 r}\big), \qquad \rho(r) = q\rho_0 + O\left(\frac{1}{r}\right) \quad \text{as } r \to \infty.$$

This solution uniquely gives rise to a finite-energy electroweak monopole solution of (2.7) *and the energy configuration is $E = \frac{4\pi}{q^2}\sin^2\theta_W M_W$ in* [2]*, where M_W is the weak energy scale.*

This theorem will be established in the next section by a shooting method.

4. Proof of existence

Using the uniqueness theorem for the solutions to the initial value problems of ordinary differential equations, we know that $0 < f(r) < 1$ for all $r > 0$. Hence we can rewrite $f'(r) + a\rho(r) f(r) = 0$ as $(\ln f(r))' + a\rho(r) = 0$. Inserting this relation into $\rho'(r) = \frac{1}{r^2}(1 - f^2(r))$ and replacing $f(r)$ with $u(r) = \ln f(r)$, we arrive at the new equation

$$r^2 u''(r) = a\big(e^{2u(r)} - 1\big), \quad 0 < r < \infty, \tag{4.1}$$

along with the updated boundary conditions

$$u(0) = 0, \qquad u(\infty) = -\infty. \tag{4.2}$$

In order to simplify our problem further, we use the Euler variable t to replace r: $r = e^t$. So our boundary value problem consisting of (4.1) and (4.2) becomes

$$\begin{cases} u''(t) - u'(t) = a\big(e^{2u(t)} - 1\big), \quad -\infty < t < \infty, \\ u(-\infty) = 0, \qquad u(\infty) = -\infty. \end{cases} \tag{4.3}$$

To solve the two-point boundary value problem (4.3), we use a dynamical shooting method. Hence we need to consider the initial value problem

$$\begin{cases} u''(t) - u'(t) = a\big(e^{2u(t)} - 1\big), \quad -\infty < t < \infty, \\ u(0) = m, \qquad u'(0) = -n. \end{cases} \tag{4.4}$$

Because we are looking for a negative-valued solution, we naturally assume $m < 0$. Under the above assumption, we will show that $n > 0$ is suitably chosen in (4.4). In this way, we may obtain a solution to (4.3). For this purpose, we set $\tau = -t$ and convert (4.4) in the half interval $-\infty < t \leqslant 0$ into the form

$$\begin{cases} u''(\tau) - u'(\tau) = a\big(e^{2u(\tau)} - 1\big), \quad \tau > 0, \\ u(0) = m, \qquad u'(0) = n. \end{cases} \tag{4.5}$$

For fixed $m<0$, we use $u(\tau;n)$ to denote the unique solution of (4.5) which is defined in its interval of existence. Define three data sets as follows

$$\beta^- = \{n\in R \mid \text{there exists } \tau>0 \text{ so that } u_\tau(\tau;n)<0\},$$
$$\beta^0 = \{n\in R \mid u_\tau(\tau;n)>0 \text{ and } u(\tau;n)\leqslant 0 \text{ for all } \tau>0\},$$
$$\beta^+ = \{n\in R \mid u_\tau(\tau;n)>0 \text{ for all } \tau\geqslant 0 \text{ and } u(\tau;n)>0 \text{ for some } \tau>0\}.$$

Lemma 4.1. *We have the disjoint union $R=\beta^-\cup\beta^0\cup\beta^+$.*

Proof. If $n\notin\beta^-$, then $u_\tau(\tau;n)\geqslant 0$ for all τ. If there exists a point $\tau_0>0$ so that $u_\tau(\tau_0;n)=0$, then $u(\tau_0;n)\neq 0$ because $u(\tau;n)=0$ is an equilibrium point of the differential equation in (4.5) which is not attainable in finite time. Using the information $u_\tau(\tau_0;n)=0$ but $u(\tau_0;n)\neq 0$ in (4.5), we see that either $u''>0$ or $u''<0$ at $\tau=\tau_0$. Hence, there is a $\tau>\tau_0$ or $\tau<\tau_0$ at which $u_\tau(\tau;n)<0$. This contradicts the assumption that $n\notin\beta^-$. Thus $u_\tau(\tau;n)>0$ for all $\tau>0$ and $n\in\beta^0\cup\beta^+$, which proves the relation $R=\beta^-\cup\beta^0\cup\beta^+$ as claimed. □

Lemma 4.2. *The sets β^+ and β^- are both open and nonempty.*

Proof. To see that β^+ is nonempty, we integrate (4.5) to get

$$u_\tau(\tau;n)=\left(n+\int_0^\tau ae^{s_1}\left(e^{2u(s_1;n)}-1\right)ds_1\right)e^{-\tau}, \tag{4.6}$$

$$u(\tau;n)=m+n\left(1-e^{-\tau}\right)+\int_0^\tau e^{-s_2}\left(\int_0^{s_2} ae^{s_1}\left(e^{2u(s_1;n)}-1\right)ds_1\right)ds_2. \tag{4.7}$$

For any fixed $\tau_0>0$, we can choose $n>0$ sufficiently large so that

$$n+\int_0^{\tau_0} ae^{s_1}\left(e^{2m}-1\right)ds_1>0, \tag{4.8}$$

$$m+n\left(1-e^{-\tau_0}\right)+\int_0^{\tau_0} e^{-s_2}\left(\int_0^{s_2} ae^{s_1}\left(e^{2m}-1\right)ds_1\right)ds_2>0. \tag{4.9}$$

Using (4.6)–(4.9), we see that there exists a $\tau_1\in(0,\tau_0)$, so that $u_\tau(\tau;n)>0$ for $\tau\in[0,\tau_1]$, $u(\tau;n)<0$ for $\tau\in[0,\tau_1)$, but $u(\tau_1;n)=0$. Hence, for any $\tau>\tau_1$, there holds

$$\begin{aligned} u_\tau(\tau;n) &\geqslant \left(n+\int_0^{\tau_1} ae^{s_1}\left(e^{2u(s_1;n)}-1\right)ds_1\right)e^{-\tau} \\ &\geqslant \left(n+\int_0^{\tau_0} ae^{s_1}\left(e^{2m}-1\right)ds_1\right)e^{-\tau}>0, \end{aligned} \tag{4.10}$$

$$u(\tau;n)>0. \tag{4.11}$$

Therefore, $n\in\beta^+$ and $\beta^+\neq\phi$.

Moreover, for $n_0\in\beta^+$, there is a $\tau_0>0$ so that $u(\tau_0;n_0)>0$. By the continuous dependence of u on the parameter n, we see that when n_1 is close to n_0, we have $u_\tau(\tau;n_1)>0$ for all $\tau\in[0,\tau_0]$ and $u(\tau_0;n_1)>0$. Using (4.10) again, we see that $u_\tau(\tau;n_1)>0$ for all $\tau>\tau_0$. So $n_1\in\beta^+$ and β^+ is open.

From (4.7) we have $(-\infty,0)\subset\beta^-$, so $\beta^-\neq\phi$. β^- is open, which is evidence. □

Lemma 4.3. *The set β^0 is nonempty and closed. Moreover, if $n\in\beta^0$, then $u(\tau;n)<0$ for all $\tau>0$.*

Proof. The first part of the lemma follows from the connectedness of R and the above lemma. In order to prove the second part, we assume that there exists a $\tau_0>0$ so that $u(\tau_0;n)=0$. Since $u(\tau;n)\leqslant 0$ for all $\tau>0$, u attains its local maximum at τ_0. In particular, $u_\tau(\tau_0;n)=0$, which contradicts the definition of β^0. □

Lemma 4.4. *For $n\in\beta^0$, we have $u(\tau;n)\to 0$ as $\tau\to\infty$.*

Proof. Since $u(\tau;n)$ increases and $u(\tau;n)<0$ for all $\tau>0$, we see that the limit

$$\lim_{\tau\to\infty} u(\tau;n)=u_\infty$$

exists and $-\infty<u_\infty\leqslant 0$. If $u_\infty<0$, then $a(e^{2u(\tau;n)}-1)<a(e^{2u_\infty}-1)<0$. Using (4.6), we have $u_\tau(\tau;n)<0$ when $\tau>0$ is sufficiently large, which contradicts the definition of β^0. □

Lemma 4.5. *The set β^0 is a single point set. Namely, the correct shooting data is unique.*

Proof. Assume that there are two points n_1 and n_2 in β^0. Let $u(\tau;n_1)$ and $u(\tau;n_2)$ be the corresponding solutions of (4.5). Then the function $\omega(\tau)=u(\tau;n_1)-u(\tau;n_2)$ solves the following boundary problem

$$\begin{cases}\omega''(\tau)+\omega'(\tau)=R'\big(\xi(\tau)\big)\omega(\tau), \quad 0<\tau<\infty,\\ \omega(0)=\omega(\infty)=0,\end{cases} \tag{4.12}$$

where $\xi(\tau)$ lies between $u(\tau;n_1)$ and $u(\tau;n_2)$ and $R'(\xi(\tau))=2ae^{2\xi(\tau)}$. Applying the maximum principle to (4.12), we have $\omega(\tau)\equiv 0$, which contradicts the assumption that $n_1\neq n_2$. □

For $n\in\beta^0$, we now consider the decay rate of $u(\tau;n)$ as $\tau\to\infty$. Since $a(e^{2u(\tau)}-1)\simeq 2au(\tau)$ near $u(\tau)=0$, we see that the linearized equation of the differential equation in (4.5) around $u(\tau)=0$ is $\theta''+\theta'=2a\theta$ whose characteristic equation has the roots $\lambda_1=-\frac{1+\sqrt{8a+1}}{2}$ and $\lambda_2=\frac{-1+\sqrt{8a+1}}{2}$. Hence, we see that for any $\varepsilon\in(0,1)$, there is a constant $C(\varepsilon)>0$ such that

$$-C(\varepsilon)e^{-\frac{1+\sqrt{8a+1}}{2}(1-\varepsilon)\tau}<u(\tau;n)<0, \quad \forall\tau\geqslant 0. \tag{4.13}$$

Now we make use of the variable $t=-\tau$, then we obtained a solution $u(t)$ of (4.4) defined in the left half of the real line, $-\infty<t\leqslant 0$, such that $u(t)<0$ for $\forall t\leqslant 0$, and

$$-C(\varepsilon)e^{\frac{1+\sqrt{8a+1}}{2}(1-\varepsilon)t}<u(t)<0, \quad \forall t\leqslant 0. \tag{4.14}$$

Next we consider the right half of the real line, $0<t<\infty$. When t is near zero, we have $u'(t)<0$ and $u(t)<0$. Inserting these into (4.4), we see $u''(t)<0$ around $t=0$. This property implies that the structure of the differential equation in (4.4) allows that $u(t)<0$, $u'(t)<0$ and $u''(t)<0$. In particular, the solution $u(t)$ exists for all $t>0$ and $u(t)$ is strictly decreasing everywhere. From (4.7), we have $u(\infty)=-\infty$. In view of Lemma 4.4, we obtained a solution of (4.3). We now strengthen our conclusion by deriving the accurate blow-up rate for $u(t)$ as $t\to\infty$.

From (4.4) we get

$$e^{-t}u'(t)=-n-a\int_0^t\big(1-e^{2u(s)}\big)e^{-s}\,ds. \tag{4.15}$$

We note that the integral on the right-hand side of (4.15) is convergent for $t\to\infty$. So we have the sharp expression

$$u'(t)=-\big(n+\sigma(t)\big)e^t \tag{4.16}$$

where $\sigma(t)=a\int_0^t(1-e^{2u(s)})e^{-s}\,ds$. We easily know $\sigma(t)$ is a bounded increasing function in $t\geqslant 0$ and $\sigma(0)=0$. Thus, we have the following expression

$$u(t)\leqslant(m+n)-ne^t, \quad \forall t\geqslant 0. \tag{4.17}$$

In other words, the function $u(t)$ blows up to $-\infty$ as fast as the function $-e^t$ as $t\to\infty$.

We need also know the asymptotic expression of $u'(t)$ as $t\to-\infty$. For this purpose, we consider the expression (4.6) in terms of the variable $\tau=-t$. Using (4.13) we see

$$u_\tau(\tau;n)=O\big(e^{-\tau}\big), \quad \tau\to\infty.$$

So we get the asymptotic estimate

$$u'(t)=O\big(e^t\big), \quad t\to-\infty. \tag{4.18}$$

Lemma 4.6. *Up to translations, $t\mapsto t+t_0$, the two-point boundary value problem* (4.3) *has a unique solution.*

Proof. Let $u_1(t)$ and $u_2(t)$ be two solutions of (4.3), then we can see that they are all negative-valued and strictly decreasing and there exists a unique point t_0 so that $u_1(0)=u_2(t_0)$. Set $u_3(t)=u_2(t+t_0)$. Then $u_1(t)$ and $u_3(t)$ are solutions of the differential equation in (4.3) and $u_1(0)=u_3(0)$. Using Lemma 4.5, we find $u_1'(0)=u_3'(0)$. Applying the uniqueness theorem for the initial value problem of an ordinary differential equation, we see $u_1(t) \equiv u_3(t)$. Namely, $u_1(t) = u_2(t + t_0)$ for all t and the lemma follows. $\square$

Similarly, we need to consider the asymptotics of $u(t)$ and $u'(t)$ as $t \to -\infty$. Using (4.13), we see $u(\tau) = O(e^{-\frac{1+\sqrt{8a+1}}{2}(1-\varepsilon)\tau})$ as $\tau \to \infty$. Inserting this into (4.5) and noting $a(e^{2u(\tau)} - 1) = O(e^{-\frac{1+\sqrt{8a+1}}{2}(1-\varepsilon)\tau})$ near $u = 0$, we have $u''(\tau) - u'(\tau) = O(e^{-\frac{1+\sqrt{8a+1}}{2}(1-\varepsilon)\tau})$. So we get $u(\tau) = O(e^{-\frac{1+\sqrt{8a+1}}{2}(1-\varepsilon)\tau})$. Note that (4.5) indicates that $u_{\tau\tau} < 0$ since $u_\tau > 0$. Hence u_τ is decreasing. So we see

$$u_\tau(\tau) < u(\tau) - u(\tau - 1) = O\left(e^{-\frac{1+\sqrt{8a+1}}{2}(1-\varepsilon)\tau}\right), \quad \forall \tau \geqslant 1. \tag{4.19}$$

In terms of $t = -\tau$, we get the improved estimates

$$u(t) = O\left(e^{\frac{1+\sqrt{8a+1}}{2}(1-\varepsilon)t}\right), \quad \text{as } t \to -\infty, \tag{4.20}$$

$$u'(t) = O\left(e^{\frac{1+\sqrt{8a+1}}{2}(1-\varepsilon)t}\right), \quad \text{as } t \to -\infty. \tag{4.21}$$

At last, we switch back to the original radial variables $r = e^t$ and the fields $(f(r), \rho(r))$. Note $\frac{du(r)}{dr} = \frac{du(r)}{dt} \cdot \frac{dt}{dr} = u'(t)e^{-t}$ and $a\rho(r) = -u'(t)e^{-t}$. Combining these relations with (4.15) and $\rho(\infty) = q\rho_0 > 0$, we have the representation

$$a = \frac{1}{q\rho_0}\left(n + a\int_0^\infty \left(1 - e^{2u(s)}\right)e^{-s}\,ds\right). \tag{4.22}$$

Lemma 4.7. *For any given $a > 0$, we can choose the initial value $m < 0$ in (4.4) so that there is a unique number $n > 0$ for which the unique solution $u(t)$ of (4.4) satisfies (4.22). Besides, such a number a depends on $m < 0$ continuously and strictly monotonically so that $a \to 0^+$ as $m \to 0^-$ and $a \to \infty$ as $m \to -\infty$.*

Proof. From Lemma 4.5 we can find that for any given $m < 0$, there exists a unique number $n > 0$ for which the unique solution $u(t)$ of (4.4) solves the two-point boundary value problem (4.3). Thus we can denote this well-defined correspondence as $n = n(m)$ and $u = u_m$.

1. We prove $n = n(m)$ is continuous with respect to $m < 0$. Let $\{m_j\}_{j=1}^\infty$ be a sequence in $(-\infty, 0)$ which converges to a number $m_0 < 0$ as $j \to \infty$. We need show that $n(m_j) \to n(m_0)$ as $j \to \infty$.

Suppose that this is not true. Then we find that there is an $\varepsilon_0 > 0$ such that $|n(m_j) - n(m_0)| \geqslant 0$ $(j = 1, 2, \ldots)$. Using Lemma 4.6, we get a sequence $\{t_j\}_{j=1}^\infty$ such that $u_{m_j}(t) = u_{m_0}(t_j + t)$ for all t. In particular, $m_j = u_{m_j}(0) = u_{m_0}(t_j)$ $(j = 1, 2, \ldots)$. Obviously, $\{t_j\}$ is a bounded sequence. Otherwise it would contradict $m_j \to m_0 < 0$ $(j \to \infty)$ and $u_{m_0}(-\infty) = 0$, $u_{m_0}(\infty) = -\infty$. By extracting a sequence if necessary, we may assume that $t_j \to t_0$ $(j \to \infty)$. So we have $n(m_j) = u'_{m_j}(0) = u'_{m_0}(t_j) \to u'_{m_0}(t_0) = n_0 \neq n(m_0)$ as $j \to \infty$. On the other hand, $m_j \to m_0$ $(j \to \infty)$ and $m_j = u_{m_j}(0) = u_{m_0}(t_j)$ $(j = 1, 2, \ldots)$ imply $t_j \to 0$ $(j \to \infty)$ since $u_{m_0}(t)$ is strictly monotone. So $t_0 = 0$ and $u'_{m_0}(t_0) = n(m_0)$. Therefore we get a contradiction.

The continuous dependence of $n(m)$ on m shows that u_m depends on m as well. Using this fact, we easily obtain the continuous dependence of the right-hand side of (4.22) on $m < 0$ because the improper integral is uniformly convergent with respect to the parameter m.

2. We claim that $n(m) \to 0$ as $m \to 0^-$. Otherwise there is a sequence $\{m_j\}_{j=1}^\infty \subset (-\infty, 0)$ and an ε_0 so that $m_j \to 0^-$ $(j \to \infty)$ but $n(m_j) \geqslant \varepsilon_0$ $(j = 1, 2, \ldots)$. Using these in (4.5) with $m = m_j$ and $n = n(m_j)$, we see that the solution may be a positive value for a slightly positive τ when j is sufficiently large which contradicts the definition of $n(m_j)$.

3. We claim that $n(m) \to \infty$ as $m \to -\infty$. Let u_0 be a fixed solution of (4.3). Using Lemma 4.6, we have that there exists a unique t_m so that $u_m(t) = u_0(t + t_m)$. Since $m = u_m(0) = u_0(t_m)$ and $u(\infty) = -\infty$, we get $t_m \to \infty$ as $m \to -\infty$. In terms of (4.17), we see $n(m) = -u'_m(0) = -u'_0(t_m) \to \infty$, as $(m \to -\infty)$.

From the above two steps we obtain that $a = a(m)$ defined by the right-hand side of (4.22) is a continuous function from $(-\infty, 0)$ to $(0, \infty)$ so that $a \to 0^+$ as $m \to 0^-$ and $a \to \infty$ as $m \to -\infty$.

4. We show that $a = a(m)$ is a strictly monotone. Let $m_1, m_2 \in (-\infty, 0)$ and $m_1 < m_2$. Denote the corresponding solutions defined above by $u_{m_1}(t)$ and $u_{m_2}(t)$. Then there is a $t_2 > 0$ such that $u_{m_2}(t_2 + t) = u_{m_1}(t)$ for all t. Since $u'(t)$ is decreasing function, we have $-n(m_1) = u'_{m_1}(0) = u'_{m_2}(t_2) < u'_{m_2}(0) = -n(m_2)$, namely $n(m_1) > n(m_2)$. Moreover since $u_{m_2}(t)$ is decreasing, we find $0 > u_{m_2}(t) > u_{m_2}(t_2 + t) = u_{m_1}(t)$, which implies that

$$\int_0^\infty \left(1 - e^{2u_{m_1}(s)}\right)e^{-s}\,ds > \int_0^\infty \left(1 - e^{2u_{m_2}(s)}\right)e^{-s}\,ds.$$

Thus we have $a(m_1) > a(m_2)$.

From $aq\rho_0 = -\lim_{t\to\infty} u'(t)e^{-t}$ we see for any $\varepsilon > 0$, there exists a $T_\varepsilon > 0$ so that when $t > T_\varepsilon$,

$$-aq\rho_0 e^t \leqslant u'(t) \leqslant (\varepsilon - aq\rho_0)e^t.$$

Integrating the above expression on $[T_\varepsilon, t]$, we get

$$u(T_\varepsilon) + aq\rho_0 e^{T_\varepsilon} - aq\rho_0 e^t \leqslant u(t) \leqslant u(T_\varepsilon) + (aq\rho_0 - \varepsilon)e^{T_\varepsilon} - (aq\rho_0 - \varepsilon)e^t. \tag{4.23}$$

Using (4.21), (4.20), (4.22), (4.15), (4.23) and with the understanding that the arbitrarily small constant $\varepsilon > 0$ is omitted in the final expressions to simplify the notation, we obtain

$$\rho(r) = O\left(r^{\frac{-1+\sqrt{8a+1}}{2}}\right), \qquad f(r) = 1 + O\left(r^{\frac{1+\sqrt{8a+1}}{2}}\right) \quad \text{as } r \to 0,$$
$$\rho(r) = q\rho_0 + O\left(\frac{1}{r}\right), \qquad f(r) = O\left(e^{-aq\rho_0 r}\right) \quad \text{as } r \to \infty.$$

The proof of lemma is now complete. □

5. Remarks

In this paper, we have obtained a series of existence results for the solutions of finite-energy electroweak monopoles. At the same time we also establish that

(i) When $q = 1$, $a = 1 + \alpha$, the asymptotic estimates obtained in our theorem are consistent with the numerical simulations in [1] and [2] near the origin.
(ii) When $a \neq b = 1$, our method may be used to obtain similar result as follows:

Theorem 5.1. *For any $a > 0$, the two-point boundary value problem* (3.3) *has a unique solution $(f(r), \rho(r))$, so that $f(r)$ is strictly decreasing and $\rho(r)$ is strictly increasing for $r > 0$. Besides, there hold the sharp asymptotic estimates*

$$f(r) = 1 + O\left(r^{\frac{1+\sqrt{8a+1}}{2}}\right), \qquad \rho(r) = O\left(r^{\frac{-1+\sqrt{8a+1}}{2}}\right) \quad \text{as } r \to 0,$$
$$f(r) = O\left(e^{-aq\rho_0 r}\right), \qquad \rho(r) = \rho_0 + O\left(\frac{1}{r}\right) \quad \text{as } r \to \infty.$$

Acknowledgment

The authors thank Professor Xiuqin Wang for helpful discussion and comments. The research of the first author was supported in part by the Natural Science Fund of Henan Education Office (2007110004) and (2008A110002).

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